# Quasi-symmetric Domains and Curvature

## §1. Introduction

In [5] we gave a characterization of quasi-symmetric domains among bounded homogeneous domains by means of socalled j-algebras, introduced by Pyatetskii-Shapiro, associated to the bounded homogeneous domain (not in a unique way). We now translate the j-algebraic conditions into more geometric conditions, involving the curvature (of the Bergman metric). For notation and results connected with j-algebras, and for description of bounded homogeneous domains and quasi-symmetric Siegel domains, we refer to [4] and [5]. We give a brief description below. The structure of j-algebras and the description of bounded homogeneous domains by them are taken from [4].

Let $\mathfrak{D}$ be an indecomposable bounded homogeneous domain. (Indecomposable: not a product of other bounded homogeneous domains.) There is a simply connected subgroup $\mathcal{G}$ of Aut $\mathfrak{D}$, with Lie algebra $\mathfrak{g}$, acting simply transitively on $\mathfrak{D}$ by (holomorphic) automorphisms. We choose a base point $o$ in $\mathfrak{D}$, and the Bergman metric on $\mathfrak{D}$ gives an invariant metric on $\mathcal{G}$. In particular we have a metric on $\mathfrak{g}$, and this metric can be written in the form $\langle X,Y\rangle = \operatorname{Re} h(X,Y) = \omega[jX,Y]$, where $h$ is the Bergman metric (at $o$), $j$ the complex structure (on the tangent space $T_o\mathfrak{D} \cong \mathfrak{g}$), and $\omega$ is a linear form on $\mathfrak{g}$. (See [4].) The invariant metric on $\mathcal{G}$ (translations of $\langle\ ,\ \rangle$ by left actions) defines the Riemannian connection $\nabla$ on $\mathcal{G} \cong \mathfrak{D}$, (the Kähler connection of the Bergman metric), and considering elements of $\mathfrak{g}$ as left invariant vector fields on $\mathcal{G}$, we have (at $o$)

$$(1)\qquad 2\langle\nabla_X Y,Z\rangle = -\langle X,[Y,Z]\rangle - \langle Y,[X,Z]\rangle + \langle Z,[X,Y]\rangle ,$$

since $\langle Y,Z\rangle$ etc. are constant on $\mathcal{G}$ . So

(2) $\quad \nabla_X Y = \frac{1}{2}\{[X,Y] - (\operatorname{ad} X)'Y - (\operatorname{ad} Y)'X\} \in \mathfrak{g} \cong T_o\mathfrak{D}$ ,

where $X,Y \in \mathfrak{g}$ and $'$ means transpose with respect to $\langle\, ,\rangle$ .

Now recall that we have the following decomposition ([4]):

$$\mathfrak{g} = \mathfrak{h} + \sum_\alpha \mathfrak{k}_\alpha = \mathfrak{l} + j\mathfrak{l} + \mathfrak{u} \quad \text{(direct vector space sums)},$$

where $\mathfrak{h} = [\mathfrak{g},\mathfrak{g}]^\perp$ (orthogonal complement with respect to $\langle\, ,\rangle$) and $[\mathfrak{g},\mathfrak{g}] = \sum_\alpha \mathfrak{k}_\alpha$ with $\mathfrak{k}_\alpha = \{X \in [\mathfrak{g},\mathfrak{g}] \mid [H,X] = \alpha(H)X \ \forall H \in \mathfrak{h}\}$ , where the <u>root</u> $\alpha$ is a linear form on $\mathfrak{h}$ . Further, if $\alpha_1,\dots,\alpha_p$ are all the roots $\alpha$ with $j\mathfrak{k}_\alpha \subset \mathfrak{h}$, then $\mathfrak{h} = j\mathfrak{k}_{\alpha_1}+\dots+j\mathfrak{k}_{\alpha_p}$, $\dim \mathfrak{h} = p$ , and, with proper enumeration, all roots are of the form $\alpha_k$, $\frac{1}{2}\alpha_k$ with $1 \leq k \leq p$ and $\frac{1}{2}(\alpha_k \pm \alpha_m)$ with $1 \leq k < m \leq p$. We have $j\,\mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)} = \mathfrak{k}_{\frac{1}{2}(\alpha_k-\alpha_m)}$ and $j\mathfrak{k}_{\frac{1}{2}\alpha_k} = \mathfrak{k}_{\frac{1}{2}\alpha_k}$ . Putting $\mathfrak{k}_k := \mathfrak{k}_{\alpha_k}$, $\mathfrak{k}_{(k,\pm m)} := \mathfrak{k}_{\frac{1}{2}(\alpha_k \pm \alpha_m)}$ and $\mathfrak{u}_k := \mathfrak{k}_{\frac{1}{2}\alpha_k}$ , then the second decomposition above is given by $\mathfrak{l} := \sum_{k=1}^{p} \mathfrak{k}_k + \sum_{1\leq k<m\leq p} \mathfrak{k}_{(k,m)}$ and $\mathfrak{u} := \sum_{k=1}^{p} \mathfrak{u}_k$ . We have $[\mathfrak{k}_\alpha, \mathfrak{k}_\beta] \subset \mathfrak{k}_{\alpha+\beta}$ ( $=\{0\}$ if $\alpha+\beta$ is no root), $\mathfrak{k}_\alpha \perp \mathfrak{k}_\beta$ for $\alpha \neq \beta$ , $\mathfrak{h} \perp \mathfrak{k}_\alpha$ , $\dim \mathfrak{k}_k = 1$ , and there is a unique non-zero element $E_k \in \mathfrak{k}_k$ with $[jE_k,E_k] = E_k$ . Also $\mathfrak{h}$ is an abelian subalgebra, $\mathfrak{l}$ an abelian ideal of $\mathfrak{g}$ , $j\mathfrak{l}$ a subalgebra, $[\mathfrak{u},\mathfrak{u}] \subset \mathfrak{l}$ , $[j\mathfrak{l},\mathfrak{u}] \subset \mathfrak{u}$ and $[\mathfrak{l},\mathfrak{u}] = \{0\}$ .

Using the properties of the $j$-algebra $\mathfrak{g}$ just described, one can easily prove (see also [1], [2])

<u>Lemma 1</u>. The action of $\nabla$ on $\mathfrak{g}$ satisfies:

1) $\nabla_H = 0$ for $H \in \mathfrak{h}$.

2) $\nabla_Y Y = |Y|^2 H_\alpha \in \mathfrak{h}$ for $Y \in \mathfrak{k}_\alpha$, where $H_\alpha$ is defined by $\alpha(H) = \langle H,H_\alpha\rangle \ \forall\, H \in \mathfrak{h}$ .

3) $\nabla_Y H = -\alpha(H)Y \in \mathfrak{k}_\alpha$ for $Y \in \mathfrak{k}_\alpha$, $H \in \mathfrak{h}$.

4) $\nabla_{E_k} E_l = \begin{cases} 0 & \text{if } k \neq l \\ jE_k & \text{if } k = l . \end{cases}$

The curvature is then given by the usual formula

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} .$$

If $Y,Z \in \mathfrak{g}$, then by the invariance of $\nabla$ also $\nabla_Y Z \in \mathfrak{g}$ etc., so we can use Lemma 1 repeatedly when calculating the curvature.

Lemma 2. We have $R(X,H) = \alpha(H)\nabla_X$ as operators on $\mathfrak{g}$, where $X \in \mathfrak{k}_\alpha$, $H \in \mathfrak{h}$.

Proof. $R(X,H) = [\nabla_X, \nabla_H] - \nabla_{[X,H]} = \nabla_{\alpha(H)X} = \alpha(H)\nabla_X$ by Lemma 1.
qed.

§2. Conditions for quasi-symmetry

Now let $Y = \sum_1^p a_k jE_k \in \mathfrak{h}$, and let $\kappa_k := |jE_k|^2 = |E_k|^2$.
Then $R(Y,jY) = \Sigma a_k R(E_k,Y) = \Sigma a_k \alpha_k(Y)\nabla_{E_k}$ by Lemma 2. Here $\alpha_k(Y) = a_k$, and, by Lemma 1, $\nabla_{E_k} E_l = \delta_{kl} jE_k$, where $\delta_{kl}$ is the Kronecker delta. So $R(Y,jY)jY = -\Sigma a_k^3 jE_k$. Hence the holomorphic sectional curvature given by $Y$ is

(3) $$K(Y) := \langle R(Y,jY)jY,Y\rangle = -\sum_1^p \kappa_k a_k^4 ,$$

where $1 = |Y|^2 = \sum_1^p \kappa_k a_k^2$.

The stationary points of $K(Y)$ on $\mathfrak{h}$ under the constraint $|Y| = 1$ are obtained from $\nabla(\Sigma \kappa_k a_k^4) = \lambda\nabla(\Sigma\kappa_k a_k^2)$, where $\nabla$ here means gradient with respect to $(a_1,\dots,a_p)$, and $\lambda$ is a Langrange multiplier. We obtain

$$2\kappa_k a_k^3 = \lambda\kappa_k a_k , \quad k = 1,\dots,p .$$

So $\lambda = 2a_k^2$ for all those $a_k$ that differ from zero, and the corresponding $|a_k|$'s are therefore equal at a stationary point. The converse is also true, hence $(a_1,\dots,a_p)$ is a stationary point (under the constraint) if and only if all non-zero $|a_k|$'s are equal. Let $(a_1,\dots,a_p)$ be a stationary point with $|a_{i_1}| = \cdots = |a_{i_t}| =: a \neq 0$ for some sequence $i_1 < \cdots < i_t$ of indices, and other $a_k$'s equal to zero, and $Y = \Sigma a_k jE_k$ be the corresponding vector. Then $1 = |Y|^2 = a^2(\kappa_{i_1}+\cdots+\kappa_{i_t})$, and hence $-K(Y) = a^4(\kappa_{i_1}+\cdots+\ _{i_t}) = (\kappa_{i_1}+\cdots+\kappa_{i_t})^{-1}$.

If $\kappa := \min\{\kappa_1,\dots,\kappa_p\}$, then

$$\max_{Y\in\mathfrak{h},|Y|=1} |K(Y)| = \kappa^{-1} \quad\text{and}\quad \min_{Y\in\mathfrak{h},|Y|=1} |K(Y)| = (\kappa_1+\cdots+\kappa_p)^{-1}.$$

Now in [5], §3,4, one condition for quasi-symmetry is

$$\text{(C)}\qquad \kappa_1 = \cdots = \kappa_p \quad (= \kappa).$$

We see that (C) is satisfied if and only if $p\kappa = \kappa_1+\cdots+\kappa_p$. Hence condition (C) is equivalent to the pinching condition

$$\text{(C')}\qquad \max_{Y\in\mathfrak{h},|Y|=1} |K(Y)| = \dim\mathfrak{h}\cdot\min_{Y\in\mathfrak{h},|Y|=1} |K(Y)|.$$

We see also that if (C') is satisfied, then the maximal value $\kappa^{-1}$ is obtained exactly at the points $\pm\kappa^{-\frac{1}{2}}jE_k$, $k = 1,\dots,p$. Further, if $X = H+\sum_\alpha X_\alpha$ with $H\in\mathfrak{h}$, $X_\alpha\in\mathfrak{k}_\alpha$, then by Lemmas 1 and 2 we have $R(X,jE_k)jE_l = \sum_\alpha R(X_\alpha,jE_k)jE_l = \sum_\alpha \alpha(jE_k)\nabla_{X_\alpha}jE_l = -\sum_\alpha \alpha(jE_k)\alpha(jE_l)X_\alpha$.

Using this, we see that

$$\text{(4)}\qquad \begin{aligned} \mathfrak{h}_{(a,b)} &= \{X\in\mathfrak{g} \mid R(X,jE_k)jE_l = \begin{cases} -X/4 & \text{if } k,l = a,b \\ 0 & \text{otherwise} \end{cases}\}, \text{ and} \\ \mathfrak{U}_a &= \{X\in\mathfrak{g} \mid R(X,jE_k)jE_l = \begin{cases} -X/4 & \text{if } k = l = a \\ 0 & \text{otherwise} \end{cases}\}. \end{aligned}$$

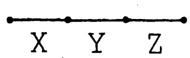
So one method to find the decomposition $[\mathfrak{g},\mathfrak{g}] = \sum_\alpha \mathfrak{k}_\alpha$ in $\mathfrak{g} = \mathfrak{h} + [\mathfrak{g},\mathfrak{g}]$, where $\mathfrak{h} = [\mathfrak{g},\mathfrak{g}]^\perp$, is as follows (if (C') is satisfied): Having $\mathcal{G}$ and hence $\mathfrak{g}$, compute $\mathfrak{h}$, then check that (C') is satisfied. Find $p = \dim \mathfrak{h}$ linearly independent points $A_1,\dots,A_p \in \mathfrak{h}$ with $|A_k| = 1$ where $M := \max_{Y\in\mathfrak{h},\,|Y|=1} |K(Y)|$ is obtained. Then, with proper enumeration, $jE_k = \pm A_k/M^{\frac12}$, and hence $\mp jA_k/M^{\frac12} = E_k = [jE_k,E_k] = [\pm A_k/M^{\frac12},\mp jA_k/M^{\frac12}] = -[A_k,jA_k]/M$. So $[A_k,jA_k] = \pm M^{\frac12} jA_k$, and the sign we get here is the same as in $jE_k = \pm A_k/M^{\frac12}$. Having found the $E_k$'s (up to enumeration), we choose the enumeration so that $a<b$ if $[jE_a,jX] = \frac12 jX$, or equivalently, $[jE_b,jX] = -\frac12 jX$, where $X \in \mathfrak{k}_{(a,b)}$, the space $\mathfrak{k}_{(a,b)}$ being defined in (4). This is possible by [4]. Letting $\alpha_k : \mathfrak{h} \to \mathbb{R}$ be the linear form $\alpha_k(\sum_l a_l jE_l) := a_k$, all roots are of the forms $\alpha_k$, $\frac12\alpha_k$ with $1 \le k \le p$, and $\frac12(\alpha_k \pm \alpha_m)$ with $1 \le k < m \le p$. And now in (4) we have $\mathfrak{k}_{(a,b)} = \mathfrak{k}_{\frac12(\alpha_a+\alpha_b)}$ and $\mathfrak{u}_a = \mathfrak{k}_{\frac12\alpha_a}$, and furthermore $j\mathfrak{k}_{(a,b)} = \mathfrak{k}_{\frac12(\alpha_a-\alpha_b)}$. So now we have the (a) j-algebra structure on $\mathfrak{g}$.

We proceed with the quasi-symmetry conditions. In [5] we defined a commutative product as follows:

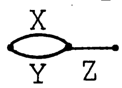
$$XY := T_X Y = T_Y X\,, \quad \text{where} \quad T_Y = \tfrac12\{\operatorname{ad} jY + (\operatorname{ad} jY)'\}\,,$$

the transpose ' being with respect to $\langle\,,\rangle$.

A necessary condition for quasi-symmetry is ([5]) that $\mathfrak{l}$ is a compact Jordan algebra with this product. It was also found that this is the case if and only if (under condition (C)) the following two conditions hold:

(A) For elements connected as 

we have $(XY)Z = X(YZ)$

(B) For elements connected as 

we have $X(YZ)+Y(XZ) = (XY)Z$,

i.e. $\quad X(YZ)+Y(XZ) = \frac{1}{4\kappa}\langle X,Y\rangle Z\,.$

The diagrams mean that $X,Y,Z \in \sum_{1\leq k<m\leq p} \mathfrak{k}_{(k,m)}$ and that for instance $X$ and $Y$ are connected as $\bullet\underset{X}{\text{—}}\bullet\underset{Y}{\text{—}}\bullet$ if $X \in \mathfrak{k}_{(k,m)}$ and $Y \in \mathfrak{k}_{(l,n)}$ with $\{k,m,l,n\}$ being a set of three different letters, and so on.

We first prove that our product coincides with the one in [1].

<u>Lemma 3</u>. We have $XY = T_X Y = -j\nabla_X Y$ for $X,Y \in \mathfrak{l}$ .

Proof. We have $-j\nabla_X Y = -\nabla_X jY = -\frac{1}{2}\{[X,jY] - (\mathrm{ad}X)'jY - (\mathrm{ad}\,jY)'X\}$ $= \frac{1}{2}\{(\mathrm{ad}\,jY)X + (\mathrm{ad}\,jY)'X\} + \frac{1}{2}(\mathrm{ad}X)'jY$ , since $j$ and $\nabla$ commute. ( $\nabla$ is a Kähler connection.) If $W$ is any element of $\mathfrak{g}$ , then $\langle(\mathrm{ad}X)'jY,W\rangle = \langle jY,[X,W]\rangle = 0$ since $[X,W] \in [\mathfrak{l},\mathfrak{g}] \subset \mathfrak{l}$ ( $\mathfrak{l}$ is an ideal), $jY \in j\mathfrak{l}$ and $\mathfrak{l} \perp j\mathfrak{l}$ . qed.

Remark: The commutativity of $XY$ follows from the fact that $\nabla_X Y - \nabla_Y X = [X,Y] = 0$ , since $[X,Y] \in [\mathfrak{l},\mathfrak{l}] = \{0\}$ .

Using the commutativity of the product, the condition in (A) is $Z(XY) = X(ZY)$ , or by Lemma 3, $-j\nabla_Z(-j\nabla_X Y) = -j\nabla_X(-j\nabla_Z Y)$ . Since $\nabla$ and $j$ commute, the condition is $\nabla_Z\nabla_X Y = \nabla_X\nabla_Z Y$ . Now $[X,Z] \in [\mathfrak{l},\mathfrak{l}] = \{0\}$ , hence $R(X,Z)Y = \nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y$ .
Hence we have

<u>Lemma 4</u>. The condition (A) is equivalent to the condition

(A') $R(X,Z)Y = 0$ for elements connected as $\bullet\underset{X}{\text{—}}\bullet\underset{Y}{\text{—}}\bullet\underset{Z}{\text{—}}\bullet$ .

Next we translate condition (B). By Lemma 3 the condition is $-j\nabla_X(-j\nabla_Y Z) - j\nabla_Y(-j\nabla_X Z) = Z(XY) = -j\nabla_Z(-j\nabla_X Y)$ , or

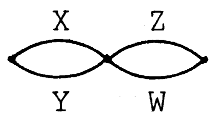
(5) $$\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z = \nabla_Z\nabla_X Y .$$

Now $\nabla_Y Z = \nabla_Z Y + [Y,Z] = \nabla_Z Y$ since $[\mathfrak{t},\mathfrak{t}] = \{0\}$, and also $[Z,X] \in [\mathfrak{t},\mathfrak{t}] = \{0\}$. Hence $R(Z,X)Y = \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y$ $= \nabla_Z\nabla_X Y - \nabla_X\nabla_Y Z$. Using (5) we see that (B) is equivalent to

(6) $\qquad R(Z,X)Y = \nabla_Y\nabla_X Z$ for $X,Y,Z$ as in (B).

Both members of this equation are contained in $\mathfrak{k}_{(k,m)}$ if $Z$ is, in general (whether or not (6) holds). This is easily checked using the description of $\nabla$, Lemma 1 and the orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \sum_\alpha \mathfrak{k}_\alpha$. So (6) is equivalent to

(7) $\qquad \langle R(Z,X)Y,W\rangle = \langle\nabla_Y\nabla_X Z,W\rangle$ for elements connected as



. We have

Lemma 5. For $S,U,V \in \mathfrak{g}$ we have $\langle\nabla_S U,V\rangle = -\langle U,\nabla_S V\rangle$.

Proof. $2\langle\nabla_S U,V\rangle = \langle[S,U],V\rangle - \langle(\mathrm{ad}\,S)'U,V\rangle - \langle(\mathrm{ad}\,U)'S,V\rangle$
$= \langle[S,U],V\rangle - \langle U,[S,V]\rangle - \langle S,[U,V]\rangle = -\langle U,[S,V]\rangle + \langle[S,U],V\rangle + \langle[V,U],S\rangle$
$= -\langle U,[S,V]\rangle + \langle U,(\mathrm{ad}\,S)'V\rangle + \langle U,(\mathrm{ad}\,V)'S\rangle = -2\langle U,\nabla_S V\rangle$.

qed.

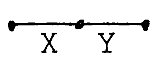
So (7) is equivalent to

(8) $\qquad \langle R(Z,X)Y,W\rangle = -\langle\nabla_X Z,\nabla_Y W\rangle$ for elements as in (7).

In particular (B) implies for the sectional curvature given by $Z$ and $X$:

(9) $\qquad K(Z,X) := \langle R(Z,X)X,Z\rangle = -|\nabla_X Z|^2$.

In general we have (under (C')):

Lemma 6. For elements connected as 

we have

$$K(Z,X) = -\frac{1}{4\kappa}|Z|^2|X|^2 + |\nabla_X Z|^2 .$$

Proof. By Lemma 5 $K(Z,X) = \langle \nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z\rangle$

$$= -\langle \nabla_X X, \nabla_Z Z\rangle + \langle \nabla_Z X, \nabla_X Z\rangle .$$

Now $\nabla_Z X = \nabla_X Z + [Z,X] = \nabla_X Z$, and by Lemma 1,
$\nabla_X X = |X|^2 H_\alpha$, $\nabla_Z Z = |Z|^2 H_\beta$ if $X \in \mathfrak{k}_\alpha$, $Z \in \mathfrak{k}_\beta$.

Here $\alpha$ is of the form $\frac{1}{2}(\alpha_l + \alpha_k)$, and $H_\alpha$ is found by Lemma 1 to be $H_\alpha = \frac{1}{2\kappa}(jE_l + jE_k)$. Similarly $H_\beta = \frac{1}{2\kappa}(jE_k + jE_m)$, and $l \neq m$. So $\langle H_\alpha, H_\beta\rangle = \frac{1}{4\kappa^2}\langle jE_k, jE_k\rangle = \frac{1}{4\kappa}$. This proves Lemma 6.

Using (9) and Lemma 6, we see that (B) implies

$$K(Z,X) = -|\nabla_X Z|^2 = -\frac{1}{4\kappa}|Z|^2|X|^2 - K(Z,X) ,$$

i.e. $K(Z,X) = -\frac{1}{8\kappa}|Z|^2|X|^2$. We have

<u>Lemma 7</u>. For elements connected as $\underset{X}{\bullet\!\!-\!\!-}\underset{Z}{\bullet\!\!-\!\!-}\bullet$ the condition (B) is equivalent to (assuming (C')) the condition

(B') $\qquad K(X,Z) = -\frac{1}{8\kappa}|X|^2|Z|^2 .$

Proof. It remains to show that (B') implies (5). Combining (B') and Lemma 6, we get (9). By polarizing in $Z$ we get $\langle R(Z,X)X, Z'\rangle = -\langle \nabla_X Z, \nabla_X Z'\rangle$, since the left hand side is symmetric in $Z$ and $Z'$. (See for instance [3], p.198.) By Lemma 5 we get $R(Z,X)X = \nabla_X \nabla_X Z$, since, as remarked earlier, both sides here lie in the same root space as $Z'$ $(Z)$ in general. Replacing $X$ by $X+Y$, we get

(10) $\qquad R(Z,X)Y + R(Z,Y)X = \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z .$

The left hand side here equals

$\nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y + \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X = 2\nabla_Z \nabla_X Y - (\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z)$, since $\nabla_Y X = \nabla_X Y + [Y,X] = \nabla_X Y$ etc.. So (10) gives

$\nabla_Z \nabla_X Y = \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z$, which is (5). qed.

The conditions (A'), (B'), (C') give the self-adjointness of a certain cone $\Omega \subset \mathcal{L}$, which is one part of the definition of quasi-symmetry. The other part concerns the relation of the map $T_X$ mentioned above with Satake's map $R_X \in \mathrm{End}_{\mathbb{C}}\mathcal{U}$, for $X \in \mathcal{L}$. The map $T_X$ is very geometric; it represents tangents through the base point $E := \sum_1^p E_k \in \Omega$. (See [5], §2.) Now let $F : \mathcal{U} \times \mathcal{U} \to \mathcal{L}_C$ be $F(u,v) := \frac{1}{4}\{[ju,v]+i[u,v]\}$. (See [4], [5]. The $\frac{1}{4}$ is unimportant.) $F$ is sesqui-linear, $\mathbb{C}$-linear in the first argument, where the complex structure on $\mathcal{U}$ is the $j$. The form $F_E : \mathcal{U} \times \mathcal{U} \to \mathbb{C}$ defined by $F_E(u,v) := \langle E, F(u,v)\rangle$, where $\langle\ ,\ \rangle$ is extended $\mathbb{C}$-bilinearly to $\mathcal{L}_{\mathbb{C}} \times \mathcal{L}_{\mathbb{C}}$, is a positive definite hermitian form on $\mathcal{U}$. (See [5].)

<u>Lemma 8</u>. We have $F_E(u,v) = \frac{1}{4}\{\langle u,v\rangle + i\langle u,jv\rangle\}$.

Proof. Let $u = \sum_1^p u_k$ with $u_k \in \mathcal{U}_k$. Then $[jE,u] = \sum_1^p [jE_k, u_k]$ $= \frac{1}{2}\sum_1^p u_k = \frac{1}{2}u$, and similarly for $v$. Hence $\langle E,[u,v]\rangle = \omega[jE,[u,v]]$ $= -\omega[u,[v,jE]] - \omega[v,[jE,u]] = \frac{1}{2}\omega[u,v] - \frac{1}{2}\omega[v,u] = \omega[u,v] = -\langle ju,v\rangle$. So $\langle E,F(u,v)\rangle = \frac{1}{4}\{-\langle j^2u,v\rangle - i\langle ju,v\rangle\} = \frac{1}{4}\{\langle u,v\rangle + i\langle u,jv\rangle\}$. qed.

The map $R_X$, for $X \in \mathcal{L}$, is defined by $2F_E(R_X u,v) = \langle X,F(u,v)\rangle$. We showed in [5] that $R_X = \frac{1}{2}\{\mathrm{ad}\,jX + (\mathrm{ad}\,jX)'\} : \mathcal{U} \to \mathcal{U}$, just as $T_X = \frac{1}{2}\{\mathrm{ad}\,jX + (\mathrm{ad}\,X)'\} : \mathcal{L} \to \mathcal{L}$. But we want to express $R_X$ in terms of $\nabla$. Let $\Phi(X) \in \mathrm{End}_{\mathbb{C}}\mathcal{U}$ be defined by $\Phi(X)u := -j\nabla_X u$.

<u>Lemma 9</u>. We have $\langle \Phi(X)u,v\rangle := \frac{1}{2}\langle X,[ju,v]\rangle$.

Proof. $\langle\Phi(X)u,v\rangle = -\langle\nabla_X ju,v\rangle = -\frac{1}{2}\langle[X,ju] - (\mathrm{ad}\,X)'ju - (\mathrm{ad}\,ju)'X, v\rangle$ $= \frac{1}{2}\langle ju,[X,v]\rangle + \frac{1}{2}\langle X,[ju,v]\rangle = \frac{1}{2}\langle X,[ju,v]\rangle$, since $[X,ju],[X,v] \in [\mathcal{L},\mathcal{U}] = \{0\}$. qed.

<u>Lemma 10</u>. We have $R_X u = -j\nabla_X u$ for $X \in \mathfrak{l}$, $u \in \mathfrak{u}$. (Compare with Lemma 3.)

Proof. By Lemmas 8 and 9, and the definitions of $F$ and $R_X$, we get $\frac{1}{2}\{\langle R_X u,v\rangle + i\langle R_X u,jv\rangle\} = 2F_E(R_X u,v) = \langle X,F(u,v)\rangle$
$= \frac{1}{4}\langle X,[ju,v]+i[u,v]\rangle = \frac{1}{2}\{\langle\Phi(X)u,v\rangle - i\langle\Phi(X)ju,v\rangle\}$
$= \frac{1}{2}\{\langle\Phi(X)u,v\rangle + i\langle\Phi(X)u,jv\rangle\}$. Since $\langle\ ,\ \rangle$ is real, we get $\langle R_X u,v\rangle = \langle\Phi(X)u,v\rangle$, proving the Lemma. qed.

Now Satake's quasi-symmetry condition (definition) is

$$\text{(Q)}\qquad \begin{aligned} T_X F(u,v) &= F(R_X u,v) + F(u,R_X v), \text{ or equivalently} \\ T_X[u,v] &= [R_X u,v] + [u,R_X v]. \qquad \text{(See [5], §4.)} \end{aligned}$$

We showed in [5], §4, that the condition (Q) is equivalent to the conditions

$(\tilde{A})$ (Q) holds for $u \in \mathfrak{u}_m$, $v \in \mathfrak{u}_a$, $X \in \mathfrak{k}_{(a,b)}$ with $a \neq m < b$,

$(\tilde{D})$ $\dim \mathfrak{u}_k$ is independent of $k$.

(In [5] the condition $(\tilde{A})$ concerned only the $(\mathrm{ad}\,jX)'$-part of $T_X$ and $R_X$. The $\mathrm{ad}\,jX$-part satisfies the condition (Q), by Leibniz.) By Lemmas 3 and 10 the condition $(\tilde{A})$ is

$$(\tilde{Q})\qquad \begin{aligned} &j\nabla_X[u,v] = [j\nabla_X u,v] + [u,j\nabla_X v] \quad \text{for } u \in \mathfrak{u}_m,\ v \in \mathfrak{u}_a, \\ &X \in \mathfrak{k}_{(a,b)} \text{ with } a \neq m < b. \end{aligned}$$

We want to rewrite this. Since $[u,v] = \nabla_u v - \nabla_v u$ (the torsion is zero, or use the expression for $\nabla$), we can write $(\tilde{Q})$ as

$$j\nabla_X[\nabla_u v - \nabla_v u] = \nabla_{j\nabla_X u} v - \nabla_v j\nabla_X u + \nabla_u j\nabla_X v - \nabla_{j\nabla_X v} u, \text{ or}$$

$$(\nabla_X\nabla_u - \nabla_u\nabla_X)v - (\nabla_X\nabla_v - \nabla_v\nabla_X)u = -\nabla_{\nabla_X ju} jv + \nabla_{\nabla_X jv} ju.$$

Now $[\mathfrak{l},\mathfrak{u}] = \{0\}$, so $R(X,u) = [\nabla_X,\nabla_u]$ etc. Hence the left hand side above can be written as $R(X,u)v - R(X,v)u = R(X,u)v+R(v,X)u = -R(u,v)X = -R(ju,jv)X$, using standard properties of the (Kählerian) curvature. Replacing $u,v$ by $-ju,-jv$, we have that the condition $(\tilde{Q})$ is equivalent to the condition

$$(\hat{Q}) \qquad R(u,v)X = \nabla_{\nabla_X u} v - \nabla_{\nabla_X v} u \quad \text{for } u \in \mathfrak{n}_m,\ v \in \mathfrak{n}_a,\ X \in \mathfrak{k}_{(a,b)} \quad \text{with } a \neq m < b .$$

Remark: As is seen (Q) is equivalent to this equation without the specification of the root spaces. But we are able to reformulate this equation (see below) only for the situation in $(\hat{Q})$, which is sufficient.

One easily sees from the expression for $\nabla$, the orthogonality of the decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha} \mathfrak{k}_{\alpha}$, and the fact that $[\mathfrak{k}_{\alpha}, \mathfrak{k}_{\beta}] \subset \mathfrak{k}_{\alpha+\beta}$, that

$$(11) \qquad \nabla_{\mathfrak{n}} \mathfrak{l} \subset \mathfrak{n}, \quad \nabla_{\mathfrak{l}} \mathfrak{n} \subset \mathfrak{n}, \quad \nabla_{\mathfrak{n}} \mathfrak{n} \subset \mathfrak{l} + j\mathfrak{l}, \quad \nabla_{\mathfrak{l}} \mathfrak{l} \subset j\mathfrak{l} .$$

Then one sees that both sides of $(\hat{Q})$ are contained in $\mathfrak{l} + j\mathfrak{l}$. So $(\hat{Q})$ is equivalent to the conditions

$$(\hat{Q}\mathfrak{l}) \qquad \langle R(u,v)X, L\rangle = \langle \nabla_{\nabla_X u} v, L\rangle - \langle \nabla_{\nabla_X v} u, L\rangle ,$$

$$(\hat{Q}j\mathfrak{l}) \qquad \langle R(u,v)X, jL\rangle = \langle \nabla_{\nabla_X u} v, jL\rangle - \langle \nabla_{\nabla_X v} u, jL\rangle$$

for $u,v,X$ as specified in $(\hat{Q})$ and $L \in \mathfrak{l}$.

The right hand side of $(\hat{Q}\mathfrak{l})$ equals $-\langle v, \nabla_{\nabla_X u} L\rangle + \langle u, \nabla_{\nabla_X v} L\rangle = -\langle v, \nabla_L \nabla_X u\rangle + \langle u, \nabla_L \nabla_X v\rangle = \langle (\nabla_X \nabla_L - \nabla_L \nabla_X) u, v\rangle$, where we used Lemma 5, the fact that the torsion is zero, the fact that $[\mathfrak{n}, \mathfrak{l}] = \{0\}$ and that $\nabla_{\mathfrak{l}} \mathfrak{n} \subset \mathfrak{n}$ by (11). Since also $[\mathfrak{l}, \mathfrak{l}] = \{0\}$, we have $R(X,L) = [\nabla_X, \nabla_L]$. Hence $(\hat{Q}\mathfrak{l})$ can be written as

$\langle R(u,v)X,L\rangle = \langle R(X,L)u,v\rangle$ , or in the standard notation, $R(L,X,u,v) = R(v,u,X,L)$ . This condition is always satisfied. (See [3], p.198.)

The right hand side of $(\hat{Q}j\ell)$ can similarly be written as

$$
\begin{aligned}
&-\langle j^2\nabla_{\nabla_X u}v, jL\rangle + \langle j^2\nabla_{\nabla_X v}u, jL\rangle \\
&= -\langle\nabla_{\nabla_X u}jv, L\rangle + \langle\nabla_{\nabla_X v}ju, L\rangle \\
&= \langle jv, \nabla_{\nabla_X u}L\rangle - \langle ju, \nabla_{\nabla_X v}L\rangle \\
&= \langle jv, \nabla_L\nabla_X u\rangle + \langle ju, j^2\nabla_L\nabla_X v\rangle \\
&= \langle(\nabla_L\nabla_X+\nabla_X\nabla_L)u, jv\rangle .
\end{aligned}
$$

Observing that $\nabla_X u = 0$ for elements as specified in $(\hat{Q})$, we can write $(\hat{Q}j\ell)$ as $\langle R(u,v)X,jL\rangle = \langle\nabla_X\nabla_L u,jv\rangle$ , or, replacing $v$ by $jv$ , as $\langle R(u,jv)X,jL\rangle = -\langle\nabla_X\nabla_L u,v\rangle$ .

Now generally $\langle R(X,L)u,v\rangle = \langle(\nabla_X\nabla_L-\nabla_L\nabla_X)u,v\rangle = \langle\nabla_X\nabla_L u,v\rangle$ for the specified $u$ and $X$ . So by the above we see that $(\tilde{A})$ is equivalent to the condition

$(\hat{A})$ $\quad \langle R(u,jv)X,jL\rangle = -\langle R(X,L)u,v\rangle$ for $u\in\mathcal{U}_m$ , $v\in\mathcal{U}_a$ ,
$X\in \mathfrak{k}_{(a,b)}$ , $L\in\ell$ , with $a \neq m < b$ .

By standard properties of the curvature the left hand side of $(\hat{A})$ equals

$$
\begin{aligned}
&-\langle R(jv,X)u,jL\rangle - \langle R(X,u)jv,jL\rangle \\
&= \langle j^2R(jv,X)u,jL\rangle - \langle jR(X,u)v,jL\rangle \\
&= \langle R(jv,X)ju,L\rangle - \langle R(X,u)v,L\rangle \\
&= -\langle R(X,ju)jv,L\rangle - \langle R(ju,jv)X,L\rangle - \langle R(X,u)v,L\rangle \\
&= -\langle R(X,ju)jv,L\rangle - \langle R(u,v)X,L\rangle - \langle R(X,u)v,L\rangle .
\end{aligned}
$$

Hence the equation in $(\hat{A})$ can be written as

$$\langle R(X,u)v,L\rangle = \langle R(X,L)u,v\rangle - \langle R(X,ju)jv,L\rangle - \langle R(u,v)X,L\rangle .$$

Now in the standard notation we have ([3], p.198) $\langle R(X,L)u,v\rangle = R(v,u,X,L) = R(L,X,u,v) = \langle R(u,v)X,L\rangle$ . Hence $(\hat{A})$ is equivalent to the condition

$(\hat{A}')$ $\langle R(X,u)v,L\rangle = -\langle R(X,ju)jv,L\rangle$ for $u \in \mathfrak{u}_m$, $v \in \mathfrak{u}_a$, $X \in \mathfrak{k}_{(a,b)}$, $L \in \mathfrak{l}$, with $a \neq m < b$ .

By (11) one sees that $R(X,u)v,\ R(X,ju)jv \in \mathfrak{l} + j\mathfrak{l}$ . Suppose the condition $(\hat{A}')$ holds. Then $\langle R(X,u)v,jL\rangle = -\langle j^2 R(X,u)v,jL\rangle = -\langle R(X,u)jv,L\rangle = \langle R(X,ju)j^2 v,L\rangle = -\langle jR(X,ju)jv,j^2 L\rangle = -\langle R(X,ju)jv,jL\rangle$. Hence we have

Lemma 11. The condition $(\tilde{A})$ is equivalent to the condition

$(\tilde{A}')$ $R(X,u)v = -R(X,ju)jv$ for $u \in \mathfrak{u}_m$, $v \in \mathfrak{u}_a$, $X \in \mathfrak{k}_{(a,b)}$, with $a \neq m < b$ .

Finally we had in [5] a condition

(D) $d_k := \sum_{l \neq k} \dim \mathfrak{k}_{\frac{1}{2}(\alpha_l+\alpha_k)}$ is independent of $k$ .

Since $\dim \mathfrak{k}_{\alpha_k} = 1$ , we can also write this as

(D) $\sum_{l=1}^{p} \dim \mathfrak{k}_{\frac{1}{2}(\alpha_l+\alpha_k)}$ is independent of $k$ .

In [5] we proved that an indecomposable bounded homogeneous domain is (biholomorphic to) a quasi-symmetric domain if and only if (A), $(\tilde{A})$, (B), (C), (D) and $(\tilde{D})$ hold; and that if $\mathfrak{u} \neq \{0\}$ , then we can skip (A).

Remarks. 1) The equation in $(\tilde{A}')$ is always satisfied for $a < b \leq m$ . (Compare [5], §4, Lemma 1.) For $m > b$ this follows by the remark after $(\hat{Q})$ from $(\hat{Q}j\mathfrak{l})$ just as above. For $m = b$ one checks that

$$R(X,u)v = \nabla_X\nabla_u v - \nabla_u\nabla_X v = \tfrac{1}{2}\nabla_X\{[u,v]-(\mathrm{ad}\,u)'v\} + \tfrac{1}{2}\nabla_u(\mathrm{ad}\,v)'X$$
$$= \tfrac{1}{2}\langle X,[u,v]\rangle\frac{1}{2\kappa}(jE_a+jE_b) + \tfrac{1}{2}j\langle X,j(\mathrm{ad}\,u)'v\rangle\frac{1}{2\kappa}(jE_a+jE_b)$$
$$+\frac{1}{4}\{[u,(\mathrm{ad}\,v)'X] - (\mathrm{ad}\,u)'(\mathrm{ad}\,v)'X - (\mathrm{ad}((\mathrm{ad}\,v)'X))'u\}$$
$$= -\frac{1}{4\kappa}\langle X,j(\mathrm{ad}\,u)'v\rangle(E_a+E_b) + \frac{1}{4\kappa}\langle X,j(\mathrm{ad}\,u)'v\rangle(E_a+E_b)$$

+ terms in $\mathbb{R}jE_a+\mathbb{R}jE_b \in \mathfrak{h}$, where we used Lemma 1 and the description of $\nabla$. Similarly $-R(X,ju)jv \in \mathfrak{h}$; hence $\langle R(X,u)v,L\rangle = -\langle R(X,ju)jv,L\rangle$ for all $L \in \mathfrak{l}$. As in the proof of Lemma 11, it follows that $R(X,u)v = -R(X,ju)jv$, or actually that $R(X,u)v = 0$ in this case.

2) One cannot assume that the equation in $(\tilde{A}')$ holds for <u>all</u> $u,v \in \mathfrak{U}$, $X \in \Sigma\,\mathfrak{k}_{(a,b)} \subset \mathfrak{l}$ in the quasi-symmetric case, since one can calculate that such an assumption would imply that

$2T_X[u,v] = [R_Xu,v] + [u,R_Xv]$ for certain $u,v$. The left hand side here is twice as big as we want.

Let us finally make the

<u>Definition</u>. A subgroup $\mathcal{G}$ of Aut $\mathcal{D}$ (= biholomorphic automorphisms) is called <u>triangular</u> if $\mathcal{G}$ is a simply connected solvable group acting simply transitively on $\mathcal{D}$, and such that $\mathrm{ad}\,X$ has only real characteristic roots for $X \in \mathfrak{g}$ := Lie algebra of $\mathcal{G}$.

Such subgroups exist ([4]), and the corresponding $\mathfrak{g}$ is a j-algebra when we choose a base point $o \in \mathcal{D}$ and identify $\mathfrak{g}$ with $T_o\mathcal{D}$.

We now translate the theorem of [5], using (C'), Lemmas 4, 7, 11, Remark 1 above and the description of how to get a j-algebra structure with root space decomposition of $\mathfrak{g}$.

<u>Theorem</u>. Let $\mathcal{D}$ be an indecomposable bounded homogeneous domain, and let $\mathcal{G}$ be a triangular subgroup of Aut $\mathcal{D}$ with Lie algebra $\mathfrak{g}$.

Choose a base point $o$ of $\mathcal{D}$ and give $\mathfrak{g}$ the structure of a j-algebra by $\mathfrak{g} \simeq T_o\mathcal{D}$, the complex structure on $T_o\mathcal{D}$ and the Bergman metric on $T_o\mathcal{D}$. Then $\mathcal{D}$ is quasi-symmetric if and only if the conditions (C'), (A'), (B'), ($\tilde{\text{A}}$'), (D), ($\tilde{\text{D}}$) below hold. If $\mathfrak{U} \neq \{0\}$, then we can skip (A'); and if $\mathfrak{U} = \{0\}$, i.e. in the case of a tube domain, then ($\tilde{\text{A}}$') and ($\tilde{\text{D}}$) are void, of course. ( R is the curvature tensor.)

(C') $\quad \max\limits_{Y\in\mathfrak{h},|Y|=1} |K(Y)| = \dim\mathfrak{h}\cdot \min\limits_{Y\in\mathfrak{h},|Y|=1} |K(Y)|$, where

$K(Y) = \langle R(Y,jY)jY,Y\rangle$ and $\mathfrak{h} = [\mathfrak{g},\mathfrak{g}]^{\perp} \subset T_o\mathcal{D}$.

(A') $\quad R(X,Z)Y = 0$ for $X,Y,Z \in \Sigma\mathfrak{k}_{(a,b)}$ connected as $\bullet\underset{X}{\text{---}}\bullet\underset{Y}{\text{---}}\bullet\underset{Z}{\text{---}}\bullet$.

(B') $\quad K(X,Z) = -\frac{1}{8}|X|^2|Z|^2\cdot \max\limits_{Y\in\mathfrak{h},|Y|=1} |K(Y)|$ for $X,Z \in \Sigma\mathfrak{k}_{(a,b)}$

connected as $\bullet\underset{X}{\text{---}}\bullet\underset{Z}{\text{---}}\bullet$, where $K(X,Z) = \langle R(X,Z)Z,X\rangle$.

($\tilde{\text{A}}$') $\quad R(X,u)v = -R(X,ju)jv$ for $u\in\mathfrak{U}_m$, $v\in\mathfrak{U}_a$, $X\in\mathfrak{k}_{(a,b)}$

with $m \neq a$. (It suffices to restrict to $a \neq m < b$ )

(D) $\quad \sum\limits_{l=1}^{\dim\mathfrak{h}} \dim\mathfrak{k}_{\frac{1}{2}(\alpha_l+\alpha_k)}$ is independent of $k$.

($\tilde{\text{D}}$) $\quad \dim\mathfrak{U}_k$ is independent of $k$.

(The root spaces in the conditions (A'), (B'), ($\tilde{\text{A}}$'), (D), ($\tilde{\text{D}}$) are found as described after condition (C') in the text for the case that (C') is satisfied.)

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